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A SPECIAL COMPLEX OF THE SECOND DEGREE AND ITS RELATION WITH THE PENCILS OF CIRCLES.

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1. Before entering upon the treatment of this problem I will make a few preliminary remarks which, although well known to the reader, may give a clearer conception of what follows.

If a straight line is given we can write the equations of its projections upon the co-ordinate planes in the following form :*

$$(1)\qquad \begin{aligned} L&=yZ-zY, \\ M&=zX-xZ, \\ N&=xY-yX, \end{aligned}$$

where x, y, z designate the current co-ordinates. The six constants L, M, N, X, Y, Z can be considered as the co-ordinates of the straight line and satisfy the relation

$$(2)\qquad LX+MY+NZ=0.$$

An algebraic complex of straight lines of the n^{th} degree is defined by an equation of the form

$$(3)\qquad F(L,\ M,\ N,\ X,\ Y,\ Z)=0,$$

F being a polynom of the n^{th} degree and homogeneous in L, M, N, X, Y, Z.

Through every point in spaces passes an infinite number of lines belonging to the complex ; they form a cone of the n^{th} order, and in every plane lies an infinite number of lines belonging to the complex and enveloping a curve of the

*We use the designation of M. Picard in his *Traite d'Analyse*, Vol. I, p. 312.

n^{th} class. Thus, a special kind of a complex of the n^{th} degree may be obtained by all the tangents of a surface of the n^{th} order, or the secants of a curve in space of the n^{th} order.

2. In our problem we define as a special complex P of the second degree the system of secants passing through a fixed conic. Every point in space determines a cone of the second order, whose elements belong to the complex and every plane intersects the conic in two points which represent a degenerated curve of the second class, whose tangents belong to the complex.

The conic itself we will describe in the following manner:

Through any two fixed points A and B of the xy-plane draw the two circles

$$(4) \qquad U_1=(x-a_1)^2+(y-b_1)^2-r_1{}^2=0,$$

$$(5) \qquad U_2=(x-a_2)^2+(y-b_2)^2-r_2{}^2=0,$$

and form the pencil of circles

$$(6) \qquad U_1-\lambda U_2=0$$

passing through A and B. At the center of every circle of the pencil erect a perpendicular to the xy-plane and equal to the radius of the circle above and below the xy-plane. The extremities of these perpendiculars lie in an equilateral hyperbola H whose plane passes through the central line of the pencil of circles and is perpendicular to tne xy-plane. The vertices of the hyperbola are equal distant from the xy-plane and lie in a perpendicular through the center of the circle with the sect AB as a diameter.*

To every point of the equilateral hyperbola belongs a circle of the pencil (6), which with the point determines a right cone whose elements all include angles of 45° with the xy-plane. We may ask what is the character of the system of lines R passing through the equilateral hyperbola and including angles of 45° with the xy-plane. For this purpose intersect the *cone-director* of these lines with the plane at infinity and establish the new complex Q consisting of all the lines passing through the intersection. As the intersection is a circle I, the complex is of the second degree and contains all the lines including angles of 45° with the plane xy.

Evidently the system R is the common solution of the complex P and Q and is therefore a *congruence*. The degree of this congruence is 6, since the degrees of P and Q are 2, and since the hyperbola H and the circle I have two points in common. Through each point in space pass two lines, and in each plane lie four lines belonging to the congruence. It is therefore of the second order and of the fourth class. Since the equilateral hyperbola H is symmetrical in regard to the xy-plane, it is easily seen that the complex and the congruence connected with it are symmetrical to the xy-plane, in other words they are reflected into themselves. It is known that through every generatrix of a congruence of straight lines pass two developable surfaces whose elements belong to

*The thought to represent points in space by circles in a plane originated with Prof. W. Fiedler, of Zurich, who applied it in his beautiful treatise on "*Cyclographie*," Teubner, Leipzig.

the congruence. In our congruence R the developable surfaces through a generatrix D are the right cone having its vertex in the hyperbola H and the hyperbolic cylinder passing through H. The focal surface of the congruence degenerates into the hyperbola H and the plane at infinity. If we designate the representation of a pencil of circles by Fiedler's method, and the complex and congruence of rays connected with it as *cyclographic*, we may now state the theorem :

The theory of the pencils of circles is identical with the theory of the cyclographic congruence.

3. To the first pencil of circles through A and B, or cyclographic congruence, we add another pencil of circles through the points C and D, or cyclographic congruence. It may be determined by two circles

$$(7) \qquad V_1 = (x-c_1)^2 + (y-d_1)^2 - s_1{}^2 = 0,$$

$$(8) \qquad V_2 = (x-c_2)^2 + (y-d_2)^2 - s_2{}^2 = 0,$$

passing through C and D and assumes the form

$$(9) \qquad V_1 - \mu V_2 = 0.$$

The corresponding congruence is obtained as in the first pencil. Designating this congruence by S and the complex through the hyperbola G which represents the pencil (9) by T we have to solve the problem to find the common part of the congruences R and S, or as these have the circle I at infinity in common, to find the common figure of the complexes P, Q, and T. Each of the hyperbolas H and G intersect the circle I in two points and as the complexes are all of the second degree, they have a ruled surface in common whose degree according to the rules of algebra is

$$2\times 2\times 2\times 2 - 2\times 2 - 2\times 2 = 8.$$

To a generatrix in this ruled surface of the eighth order can be found one in the same surface symmetrical to the first in regard to the xy-plane. Hence the whole surface is symmetrical to the xy-plane and as it contains two double generatrices through the circular points of the circle I, it intersects the xy-plane in a bicircular curve of the fourth order. Every generatrix of the surface intersects the xy-plane in a point of the curve and includes an angle of 45° with the xy-plane.

Through each generatrix pass four developable surfaces, two hyperbolic cylinders and two cones of the second order. These cones are tangent to each other and intersect the xy-plane in two tangent circles. As these circles always pass through A, B and C, D and as their point of tangency lies in the above curve we have the theorem :

The locus of the points of tangency of each two tangent circles of two pencils of circles is a bicircular curve of the fourth order.

Figure 1 will show the relation of these pencils in the case that each two circles are tangent.

4. We will now take another view of the problem. For fixed values of λ and μ the equations of two circles respectively belonging to the pencil (6) and (9) may be written

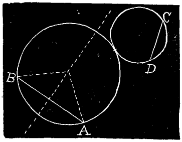

Fig. 1.

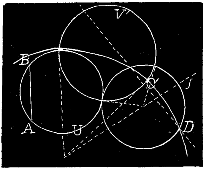

Fig. 2.

$$(10) \qquad x^2+y^2-2\frac{a_1-\lambda a_2}{1-\lambda}x-2\frac{b_1-\lambda b_2}{1-\lambda}y+\frac{M_1-\lambda M_2}{1-\lambda}=0,$$

$$(11) \qquad x^2+y^2-2\frac{c_1-\mu c_2}{1-\mu}\;-2\frac{d_1-\mu d_2}{1-\lambda}y+\frac{N_1-\mu N_2}{1-\mu}=0,$$

where $\qquad M_1=a_1{}^2+b_1{}^2-r_1{}^2,\ M_2=a_2{}^2+b_2{}^2-r_2{}^2,$

$$N_1=c_1{}^2+d_1{}^2-s_1{}^2,\ N_2=c_2{}^2+d_2{}^2-s_2{}^2.$$

The condition that the circle (10) is orthogonal to the circle (11) is

$$2\frac{a_1-\lambda a_2}{1-\lambda}\cdot\frac{c_1-\mu c_2}{1-\mu}+2\frac{b_1-\lambda b_2}{1-\lambda}\cdot\frac{d_1-\mu d_2}{1-\mu}-\frac{M_1-\lambda M_2}{1-\lambda}-\frac{N_1-\mu N_2}{1-\mu}=0.$$

It is now possible to determine the co-efficients of this equation such that for variable parameters the pencils (10) and (11) are projective. In this case each two corresponding circles are orthogonal.

Evidently we have to put $\mu=\lambda$, which after some reductions gives for the equation of condition

$$[2a_1c_1+2b_1d_1-M_1-N_1]-\lambda[2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2]$$
$$+\lambda^2[2a_2c_2+2b_2d_2-M_2-N_2]=0. \qquad (12)$$

This indicates that in the first place the circles (4) and (7), and (5) and (8) must be orthogonal. Secondly, for every value of λ there must be

$$2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2=0.$$

This equation is satisfied if the circles (5) and (7), and (4) and (8) are orthogonal. In this case the pencils (6) and (9) are said to be conjugate pencils of circles. Every circle of the one pencil is orthogonal to every other circle of the other. But in equation (12) we do not desire to change the points A, B, C, D. Thus, the equation can be satisfied by changing the radii of the circles (4), (5), (7), (8), which gives for the solutions of M_1, M_2, N_1, N_2 three equations with four unknown quantities. We can however fix one of the circles without altering the final result. This implies the theorem :

Two pencils of circles can be made projective in one and only one way such that corresponding circles in the projectivity are orthogonal.

The product of these projective pencils is a bicircular curve of the fourth order, as it is well known. In figure 2, we consider the two pencils of circles through A and B, and C' and D', where C' and D' are assumed to be imaginary and on the line l and in these pencils two orthogonal circles U and V' intersecting each other in two points J and J'. In these points draw tangent circles to U having their centers on l. These circles are orthogonal to V' and intersect each other in two fixed points C and D, *i. e.*, they belong to the conjugate pencil of circles of the pencil through C' and D'. Whence the general theorem:

The locus of the points of tangency of each two tangent-circles of two pencils of circles is a bicircular curve of the fourth order. The same curve is also produced by one of the pencils and the projective conjugate pencil of the other pencil.

Under the given conditions the equation of the curve may be written $U_1V_2'-U_2V_1'=0$.

It is easily seen that this curve passes through the four points A, B, C, D and as stated in the theorem contains the circular points at infinity as double-points.

5. Without entering into further details on the nature of this curve it may be mentioned that there exists an interesting connection between this curve and the circular curves of the third order if these are considered as locii of points from which two sects AB and CD appear under the same angle. An analogon exists in space, the discussion of which however goes over the limits of this article. A paper on this subject by the author was read in the January session of the Kansas Academy of Science and will appear in the next volume of the transactions of this Academy.

PROBLEMS.

1. Given n straight lines in a plane. Another straight line in this plane revolves about a fixed point and in every position intersects the n lines

in n points. These points determine n "sects" measured from the fixed point and their algebraic sum represents a point on the revolving line. What is the curve which this point describes?

2. Find a geometrical construction for the following problem: Given the distances AO, BO, CO of the points A, B, C of an equilateral triangle from a fixed point O. Construct the equilateral triangle, or triangles satisfying these conditions.

3. What is the locus of the points from which any two sects in space AB and CD (not in the same plane) appear under the same constant angle?

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from April Number.]

PROPOSITION XXIV. *The same hypothesis remaining: I say the four angles together (Fig. 27.) of the quadrilateral KDHK nearer the base AB are less (in hypothesis of acute angle) than the four angles together of the quadrilateral KHLK more remote from the same base; and indeed this is so, whether those two AX, BX somewhere at a finite distance meet toward the parts of the point X; or never meet one another; but toward those parts either ever more mutually approach each other, or somewhere receive a common perpendicular, after which of course (in accordance with Cor. II. of the preceding proposition) toward the same parts they begin mutually to separate.*

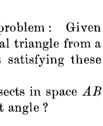

Fig. 27.

PROOF. Here however we suppose the portions KK assumed to be mutually equal. Since therefore (from the preceding) the side DK is greater than the side HK, and similarly HK greater than the side LK, the portion MK in HK is assumed equal to LK, and in DK the portion NK equal to HK; and MN, MK, LK are joined, truly the intermediate point K with the point L, and the point K near to the point B with the point M.

Now I proceed thus.

Since indeed the sides of the triangle KKL (I make beginning always from the point K nearer the point B) are equal to the sides of the triangle KKM, and the included angles equal, as being right, equal also will be (from Eu. I. 4) the bases LK, MK, and likewise equal the angles which correspond mutually,

at these bases, indeed the angle KLK to the angle KMK, and the angle LKK to the angle MKK. Therefore equal also are the remainders NKM and HKL. Wherefore, since the sides NK, KM of the triangle NKM are equal in the same way to the sides HK, KL of the triangle HKL, equal also will be (from the same Eu. I. 4) the bases NM, HL, the angles KNM, KHL, and finally the angles KMN, KLH. But in the preceding triangles are already proved equal the angles KLK, KMK. Therefore the whole angle NMK is equal to the whole angle HLK.

Wherefore, since all angles at the points K are right, it follows manifestly all four angles together of the quadrilateral $KNMK$ are equal to all four angles together of the quadrilateral $KHLK$.

But since the two angles together at the points N and M in the quadrilateral $KNMK$ are greater, in hypothesis of acute angle, than the two angles together (from Cor. after P. XVI) at the points D and H in the quadrilateral $NDHM$, or the quadrilateral $KDHK$, the consequence thence is, that (the common right angles at the points K being added) the four angles together of the quadrilateral $KNMK$, or the quadrilateral $KHLK$ are greater (in hypothesis of acute angle) than the four angles together of the quadrilateral $KDHK$.

Quod erat demonstrandum.

COROLLARY.

But it ought here opportunely to be observed, nothing will fail in the argument made, although the angle at the point L is assumed right, together with hypothesis of acute angle. For still that common perpendicular LK would be less (from Cor. I. after II of this) than the other perpendicular HK, from which therefore still a portion MK could be assumed equal to the aforesaid LK.

Which standing, it follows that no hindrance can intervene.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from April Number.]

PRIMITIVE GROUPS OF TWO, THREE, AND FOUR, LETTERS.

Since all of these must contain substitutions of the form a_1a_2 or of the form $a_1a_2a_3$ they must all contain the symmetric group. The following is therefore a complete list:

Degree.	Order.	Group.
2	2	(ab)
3	3	(abc)
	6	(abc)all
4	12	$(abcd)$pos
	24	$(abcd)$all.

Primitive Groups of Five Letters.

All the transitive groups of this degree must be primitive and there must be one regular group, viz:

(1) $(abcde)$.

The lowest order of any other possible primitive group is 10. Such a primitive group would contain five subgroups of the form $(ab.cd)$ and therefore four substitutions of degree 5. All the substitutions of degree 5 whose powers do not contain a substitution of the type a_1a_2 are of the type $a_1a_2a_3a_4a_5$. Hence all the substitutions of degree 5 of a primitive group which is not the symmetric group must be of the given type.

If a primitive group of order 10 exists we may therefore assume that it contains

$$(abcde)$$

and some substitutions of the form $ab.cd$. These substitutions are all equal to

$$(abcde)S$$

where S is any one among them. They therefore transform the substitutions of $(abcde)$ into the same power. This cannot be the first power for a substitution consisting of a single cycle can be transformed into the first power only by its own powers. If we represent this power by α and observe that the product of two of these supstitutions is equal to some substitution in $(abcde)$ we have*

$$\alpha^2 \equiv 1 \pmod 5), \ (1 < \alpha < 5).$$

Since this has only one solution it follows that there is only one group of order 10. We may find the substitutions by writing the fourth power of $abcde$ under $abcde$, thus,

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*If s_1 and s_2 transform s into s^a we have

$$s_1^{-1}ss_1 = s^a$$

$$s_2^{-1}s_1^{-1}ss_1s_2 = s_2^{-1}s^a s_2 = s_2^{-1}ss_2 . s_2^{-1}ss_2 \ \ldots\ldots\ldots\ldots \alpha \text{ times}$$

$$= s^a . s^a . s^a \ \ldots\ldots\ldots\ldots \alpha \text{ times}$$

$$= s^{\alpha^2}$$

$$\begin{matrix} abcde & abcde & abcde & abcde & abcde \\ aedcb & baedc & cbaed & dcbae & edcba. \end{matrix}$$

The required substitutions are

$$be.cd \qquad ab.ce \qquad ac.de \qquad ad.bc \qquad ae.bd$$

[We might evidently have obtained all of these by multiplying one into $(abcde)$]. Hence the group of order 10 is

(2) $$(abcde)(ab.ce)=(abcde)_{10}.$$

All the substitutions that transform $(abcde)$ into itself form a group. There are five substitutions that transform the substitutions of $(abcde)$ into their first power, therefore there must be five that transform them into each of their other powers. We thus obtain a group of order 20 which is generated by $abcde$ and some substitution $bced$ which transforms this into its second power. We have therefore

(3) $$(abcde)(bced)=(abcde)_{20}.$$

There cannot be more than one of this order because each would have to contain five conjugate subgroups of one of the two types

$$(abcd)_4,\ (abcd)$$

and therefore only one subgroup (necessarily self conjugate) of the type

$$(abcde).$$

This may be supposed to be the same in all of the groups; but there is only one set of twenty substitutions that transform this into itself. The groups are therefore identical.

For all the other possible orders the subgroups of degree 4 would contain either a substitution of the type ab or one of the type abc. Hence all the other primitive groups are the alternating and the symmetric group. The following is a complete list of the primitive groups of degree five.

Order.	Group.
5	$(abcde)$
10	$(abcde)_{10}$
20	$(abcde)_{20}$
60	$(abcde)$pos
120	$(abcde)$all

These groups could also have been found in the following manner, without employing the groups of a lower degree. We know that there is one group of each of the three classes—regular, alternating and symmetric. We know also that the order of each of the other primitive groups exceeds five and that they do not contain any substitutions of either of the two types

ab *abc*

Hence they can contain only substitutions of the fourth and fifth degrees together with unity.

Since the average number of letters in all the substitutions of these groups must be four each group can contain only four substitutions of the fifth degree. The only type of substitutions of the fifth degree which can be used is

abcde.

All these primitive groups may therefore be supposed to contain

(*abcde*)

as a self-conjugate subgroup and to be subgroups of the group of order 20 which contains all the substitutions that transform (*abcde*) into itself.

Any negative substitution of this group together with (*abcde*) generates the entire group, the only subgroup besides the group itself and (*abcde*) must therefore consist of the positive substitutions of the group. Hence there are only two primitive groups of degree five in addition to the regular, alternating, and symmetric groups. The generating substitutions of these groups are evident.

Primitive Groups of Six Letters.

There is no regular group. If there were a group of order 30 it would contain 24 substitutions of the type *abcde* and five substitutions of degree six. These five substitutions would generate a regular group; for only one of them could replace a given letter by any required letter since there are four of the form *abcde* which perform this operation, and therefore the product of any two must be of degree six or it must be unity.

[To be Continued.]

SIMULTANEOUS QUADRATIC EQUATIONS.

By I. H. BRYANT, Instructor of Mathematics, Waco High School, Waco, Texas.

This discussion is restricted to the special cases of simultaneous quadratic equations of n variables which always admit of solution. It is assumed that solutions are always possible :

(1) When there is one equation of the second degree and one variable.

(2) When all equations except one are of the first degree.

Let $q, q_1 \ldots\ldots \ldots\ldots q_n$—terms of the second degree.

Let $p, p_1 \ldots\ldots\ldots\ldots p_n$—terms of the first degree.

Let $k, k_1 \ldots\ldots\ldots\ldots k_n$=absolute terms.

Let $l, l_1 \ldots\ldots \ldots\ldots l_n$=absolute terms.

Let m=a constant factor.

Let $x, x_1 \ldots\ldots\ldots\ldots x_n$=the variables.

Let $v_1, v_2 \ldots\ldots\ldots\ldots v_n$=the variables when the equations are transformed.

CASE 1. When one equation is general, and the rest are of the first degree, or reducible to the first degree; *i. e.* when they assume any of the following forms :

(1) $(p+k)^n=0.$

(2) $(p+k)(p_1+k_1)\ldots\ldots(p_n+k_n)=0.$

(3) $(p+k)^n+m(p_1+k_1)^n=0.$

(4) $(p+k)^{2n}+m(p+k)^n+l=0.$

In the next four cases one or more of the equations may assume the above forms instead of the forms of these cases.

CASE 2. When each equation can be resolved into two factors of the first degree and an absolute term and when one of these factors is common to all equations.

$$\left\{\begin{array}{c}(p+k)(p_1+k_1)+l_1=0. \\ (p+k)(p_2+k_2)+l_2=0. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (p+k)(p_n+k_n)+l_n=0.\end{array}\right\}$$

Eliminate the common factor. There are now $n-1$ equations of the first degree.

CASE 3. When each equation can be resolved into two factors of the first degree and an absolute term and when each factor occurs in two equations.

As in the previous case, $n-1$ equations of the first degree can be obtained.

$$\left\{\begin{array}{l}(p_1+k_1)(p_2+k_2)+l_1=0.\\(p_2+k_2)(p_3+k_3)+l_2=0.\\\dots\dots\dots\dots\dots\\(p_n+k_n)(p_1+k_1)+l_n=0.\end{array}\right\}$$

Case 4. When like terms save the terms in which one variable occurs, are equal, or can be made equal in all equations. The terms can be made equal when the co-efficients of like terms are proportional in all equations, or when they are all similar and but one occurs in each equation. By eliminating these equal terms we can obtain $n-1$ equations in which the same variable will occur in each term. By dividing by this variable, we can reduce each of these equations to the first degree.

Case 5. When like terms of the second degree are equal, or can be made equal, in all equations. Like terms can be made equal when they meet with the requirements indicated in Case 4. Eliminate the terms of the second degree. There are now $n-1$ equations of the first degree.

Case 6. When the equations are homogeneous and like terms save those in which one variable occurs and the absolute term, are equal, or can be made equal in all equations. The requirements for making like terms equal are given in Case 4.

$$\left\{\begin{array}{l}p_1x+q+k_1=0.\\p_2x+q+k_2=0.\\\dots\dots\dots\dots\\p_nx+q+k_n=0.\end{array}\right\}$$

Let $x_1=v_1x$, $x_2=v_2x\dots\dots x_n=v_nx$. Eliminate x^2. We now have $n-1$ equations with $n-1$ variables which meet with the requirements of Case 5.

Case 7. When two equations are homogeneous and the rest are of the first degree, or reducible to the first degree, with no absolute term. Eliminate all except two variables from the two homogeneous equations by means of the equations of the first degree. These two equations will then be homogeneous and will fall under Case 6.

Case 8. When the terms containing one variable are equal, or can be made equal, in all equations and the remaining terms meet with the requirements of Case 6. Eliminate the terms containing this variable. We now have $n-1$ equations and $n-1$ variables which fall under Case 6.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{18}{25}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{6}{7}$ weeks longer than 660 sheep to eat up 9 acres.

In what time would an ox, a colt and a sheep together eat up an acre of the pasture on the supposition that 589 sheep eat as much in a week as 6 oxen and 11 colts? By Arithmetic, if possible.—Hunter's Arithmetic. (Unsolved in *School Visitor.*)

II. Solution by Henry Heaton, M. S., Atlantic, Iowa.

Since 6 oxen $=10$ colts, 1 ox $=1\frac{2}{3}$ colts, and 6 oxen and 11 colts $=21$ colts $=589$ sheep. $\therefore$ 1 colt $=28\frac{1}{21}$ sheep and 1 ox $=1\frac{2}{3}\times28\frac{1}{21}$ sheep $=46\frac{4}{63}$ sheep.

10 oxen and 6 colts $=22\frac{2}{3}$ colts, eat 8 acres of grass in the same time that $\frac{1}{8}$ of $22\frac{2}{3}$ colts or $2\frac{5}{6}$ colts eat 1 acre, and $3\frac{1}{3}$ colts eat an acre in the same time that 10 colts eat 3 acres. Hence $3\frac{1}{3}$ colts eat an acre in $\frac{18}{25}$ the time that $2\frac{5}{6}$ colts eat it. In $\frac{18}{25}$ the time $3\frac{1}{3}$ colts eat as much grass as $\frac{18}{25}$ of $3\frac{1}{3}$ colts or $2\frac{2}{5}$ colts would eat it in the full time. The difference between $2\frac{2}{5}$ colts and $3\frac{1}{3}$ colts is $\frac{13}{30}$ of a colt. The difference in the grass eaten by them is $\frac{7}{25}$ of the growth. Hence $\frac{13}{30}$ of a colt eats $\frac{7}{25}$ of the growth. Hence to eat all the growth will require $\frac{25}{7}$ of $\frac{13}{30}$ of a colt or $\frac{65}{42}$ of a colt $=\frac{65}{42}$ of $28\frac{1}{21}$ sheep $=43\frac{359}{882}$ sheep. To eat the growth on 9 acres will require 9 times $43\frac{359}{882}$ sheep $=390\frac{65}{98}$ sheep. $600-390\frac{65}{98}=209\frac{33}{98}$. $660-390\frac{65}{98}$ $269\frac{33}{98}$. Hence it will require $209\frac{33}{98}$ sheep $2\frac{6}{7}$ weeks longer to eat the original grass on 9 acres than it will $269\frac{33}{98}$ sheep to eat the same. Hence $209\frac{33}{98}$ sheep eat in the $2\frac{6}{7}$ weeks what the 60 other sheep eat in the first part of the time. Hence this time is $209\frac{33}{98}\times2\frac{6}{7}$ weeks $\div60=9\frac{1993}{2058}$ weeks. Hence it will take $269\frac{33}{98}$ sheep $9\frac{1993}{2058}$ weeks to eat the original grass on 9 acres. To eat 1 acre will require them $1\frac{1993}{18522}$ weeks.

An ox, a colt, and a sheep $=75\frac{50}{63}$ sheep.

If $75\frac{50}{63}$ sheep were eating on one acre, $43\frac{359}{882}$ sheep would eat the growth leaving $32\frac{341}{882}$ sheep to eat the original grass. If it require $269\frac{33}{98}$ sheep $1\frac{1993}{18522}$ weeks to do this, it will require $32\frac{341}{882}$ sheep $(269\frac{33}{98}\div32\frac{341}{882})\times1\frac{1993}{18522}$ weeks $=9\frac{3500813}{17664596}$ weeks.

58. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two men, A and B, in Boston, hire a carriage for \$25, to go to Concord, N. H., and back, the distance being 72 miles, with the privilege of taking in three more persons. Having gone 20 miles, they take in C; at Concord, they take in D; and when within 30 miles of Boston, they take in E. How much shall each man pay? [From *Greenleaf's National Arithmetic.*]

I. Solution by H. C. WHITAKER, A. M., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

If we denote taking one person one mile by a person-mile, then the total person-miles was 514 and the cost of each of them was 4.8638 cents ; the cost of taking A and B 144 miles was \$7 each ; the cost of taking C 124 miles was \$6.03 ; the cost of taking D 72 miles was \$3.50, and the cost of taking E 30 miles was \$1.46.

II. Solution by F. M. McGAW, Bordentown, New Jersey ; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas, and H. C. WILKES, Skull Run, West Virginia.

Five men ride 30 miles ; four, 42 miles ; three, 52 miles ; and two, 20 miles.

$\therefore$ E pays for $\frac{1}{5}$ of 30 = 6 miles.

D pays for $\frac{1}{5}$ of $30+\frac{1}{4}$ of $42=16\frac{1}{2}$ miles.

C pays for $\frac{1}{5}$ of $30+\frac{1}{4}$ of $42+\frac{1}{3}$ of $52=33\frac{5}{6}$ miles.

B pays for $\frac{1}{5}$ of $30+\frac{1}{4}$ of $42+\frac{1}{3}$ of $52+\frac{1}{2}$ of $20=43\frac{5}{6}$ miles.

A pays for $\frac{1}{5}$ of $30+\frac{1}{4}$ of $42+\frac{1}{3}$ of $52+\frac{1}{2}$ of $20=43\frac{5}{6}$ miles.

$144:43\frac{5}{6}=\$25:\$7.609\frac{103}{108}$, share of each A and B.

$144:33\frac{5}{6}=\$25:\$5.873\frac{91}{108}$, share of C.

$144:16\frac{1}{2}=\$25:\$2.864\frac{7}{12}$, share of D.

$144:6=\$25:\$1.041\frac{2}{3}$, share of E.

III. Solution by A. P. REED, A. M., Clarence, Missouri, and J. C. CORBIN, Pine Bluff, Arkansas.

144 miles = distance A rides, 144 miles = distance B rides, 124 miles = distance C rides, 72 miles = distance D rides, and 30 miles = distance E rides.

They should each pay in proportion to the distance each rides. Hence

$\frac{144}{514}$ of $\$25=\$7.00\frac{100}{257}$ = amount A should pay.

$\frac{144}{514}$ of $\$25=\$7.00\frac{100}{257}$ = amount B should pay.

$\frac{124}{514}$ of $\$25=\$6.03\frac{29}{257}$ = amount C should pay.

$\frac{72}{514}$ of $\$25=\$3.50\frac{50}{257}$ = amount D should pay.

$\frac{30}{514}$ of $\$25=\$1.45\frac{235}{257}$ = amount E should pay.

[NOTE. Greenleaf gives the answers as obtained in the second solution. But we think it is best to solve the problem on the principle that each pay in proportion to the distance he rides. This principle prevails in practice at the present time and is just in its application. EDITOR.]

PROBLEMS.

60. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

A pipe 1 foot long and $\frac{27}{30}$ inch in diameter has a half-inch orifice and weighs $1\frac{3}{4}$ pounds. What is the diameter of a pipe of the same length and orifice, but weighing 41 ounces ?

61. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Insured my store for a/bth $=\frac{3}{4}$th part of its value, at $r=1\frac{1}{4}$ per cent.; but soon afterward the store was burned down, and my loss over the insurance was $\$L=\$4150$. What was the value of my store ?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

58. Proposed by D. G. DORRANCE, Jr., Camden, Oneida County, New York.

Sum the series 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, etc., to n terms; also what is the n^{th} term ?

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee, and Prof. P. S. BERG, Larimore, North Dakota.

The series is evidently made up as follows from the different rows in Pascal's Triangle, beginning three farther to the right every time; thus,

a.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1 ......
b.				1	2	3	4	5	6	7	8	9	10	11	12	13 ......
c.							1	3	6	10	15	21	28	36	45	55
d.										1	4	10	20	35	56	84
e.													1	5	15	35
f.																1
	1,	1,	1,	2,	3,	4	6,	9,	13,	19,	28,	41,	60,	88,	129,	189, etc.

The n^{th} term of (a) is 1; the $(n-3)^{th}$ term of (b) is $n-3$; the $(n-6)^{th}$ term of (c) is $\frac{(n-5)(n-6)}{\underline{|2}}$; the $(n-9)^{th}$ term of (d) is $\frac{(n-7)(n-8)(n-9)}{\underline{|3}}$; and the $(n-12)^{th}$ term of (e) is $\frac{(n-9)(n-10)(n-11)(n-12)}{\underline{|4}}$; and so on. Hence the n^{th} term of the original series is composed of the sum of the above different terms; *i. e.*

$$1+(n-3)+\frac{(n-5)(n-6)}{\underline{|2}}+\frac{(n-7)(n-8)(n-9)}{\underline{|3}}+\frac{(n-9)(-10)(n-11)(n-12)}{\underline{|4}}$$

$+......$ Also, the sum of n terms of (a) is n; of $(n-3)$ terms of (b) is $\frac{(n-3)(n-2)}{\underline{|2}}$; of $(n-6)$ terms of (c) is $\frac{(n-6)(n-5)(n-4)}{\underline{|3}}$; and the sum of $(n-9)$ terms of (d) is $\frac{(n-9)(n-8)(n-7)(n-6)}{\underline{|4}}......$ and hence $S=n+$

$$\frac{(n-3)(n-2)}{\underline{|2}}+\frac{(n-6)(n-5)(n-4)}{\underline{|3}}+\frac{(n-9)(n-8)(n-7)(n-6)}{\underline{|4}}+......$$

Also solved by *B. F. YANNEY* and *G. B. M. ZERR*.

59. Proposed by DAVID EUGENE SMITH, Ph. D., Professor of Mathematics in Michigan State Normal School, Ypsilanti, Michigan.

Prove that the product of the n n^{th} roots of 1 is $+1$ or -1 according as n is odd or even. Prove and generalize, for the n n^{th} roots of m.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

$$\text{I.}\quad (1)^{\frac{1}{n}}=\cos\frac{2m\pi}{n}+\sqrt{-1}\,\sin\frac{2m\pi}{n}.$$

The several roots are: $\varepsilon^{0\sqrt{-1}}$, $\varepsilon^{(2\pi/n)\sqrt{-1}}$, $\varepsilon^{(4\pi/n)\sqrt{-1}}$, ...$\varepsilon^{[2(n-1)\pi/n]\sqrt{-1}}$.

$\therefore$ Product$=\varepsilon^{(n-1)\pi\sqrt{-1}}=\cos(n-1)\pi+\sqrt{-1}\,\sin(n-1)\pi=\pm1$.

If n is even product is negative; if n is odd product is positive.

II. Let $m=x+y\sqrt{-1}$.

$$\text{Then } (x+y\sqrt{-1})^{\frac{1}{n}}=(\sqrt{x^2+y^2})^{\frac{1}{n}}\left[\cos\left(\frac{2m\pi+\theta}{n}\right)+\sqrt{-1}\,\sin\left(\frac{2m\pi+\theta}{n}\right)\right],$$

where $\theta=\tan^{-1}\dfrac{x}{y}$.

$$R_1=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{(\theta/n)\sqrt{-1}},$$

$$R_2=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{[(2\pi+\theta)/n]\sqrt{-1}},$$

$$R_3=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{[(4\pi+\theta)/n]\sqrt{-1}},$$

....

$$R_n=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{\{[2(n-1)\pi+\theta]/n\}\sqrt{-1}}.$$

$$P=\sqrt{x^2+y^2}\,\varepsilon^{[(n-1)\pi+\theta]\sqrt{-1}}=\left[\cos\left((n-1)\pi+\theta\right)+\sqrt{-1}\,\sin[(n-1)\pi+\theta]\right]$$

$$\sqrt{x^2+y^2}=\pm\sqrt{x^2+y^2}\left[\cos\theta+\sqrt{-1}\,\sin\theta\right].$$

II. Solution by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

I. $x^n-1=0$ is the equation from which are derived the n n^{th} roots of 1. Now, since 1 in the equation is negative there is one positive real root and $(n-1)$, if any, imaginary roots, if n is odd; and there is one positive real root, one negative real root, and $(n-2)$, if any, imaginary roots, if n is even. $\therefore$ Since the

imaginary roots occur in conjugate pairs, and since the product of any two conjugate imaginaries is a positive real number, the sign of the product of the n n^{th} roots of 1 when n is odd, is $+$; and when n is even, $-$.

Furthermore, since the successive powers of the first imaginary root of 1, from the 1st to the n^{th}, give us all the n^{th} roots of 1, therefore, if we denote the first imaginary root by ω, we shall have as the product of the n n^{th} roots, $\omega.\omega^2.\omega^3 \ldots\ldots \omega^n = \omega^{n\frac{n+1}{2}}$. But $\omega^n=1$. $\therefore$ $\omega^{n\frac{n+1}{2}}=+1$ when n is odd ; and ±1, when n is even. But of these last two signs, $-$ must be chosen, for reasons assigned in the preceding paragraph.

II. That the theorem is true in general for the n n^{th} roots of m, is made evident when we remember that the n n^{th} roots of any number may be found by multiplying any one of the n^{th} roots of such number by the different n^{th} roots of 1. For then, we would have $m^{\frac{1}{n}}\times\omega.m^{\frac{1}{n}}\times\omega^2 \ldots\ldots m^{\frac{1}{n}}.\omega^n = m\omega^{n\frac{n+1}{2}} = +m$ or $-m$, according as n is odd or even, as shown above.

Also solved by *COOPER D. SCHMITT.*

ERRATA. In numerator of the expression, in next to last line, on page 115 of last issue, for "Ra" read $\frac{Ra}{r}$; on page 117, line 4, for "$s(s-2a_2)$" read $s(s-2a_3)$; and in "Errata," for "last issue" read February issue. Also problems numbered 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, should be Nos. 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, respectively.

PROBLEMS.

68. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Sum to n terms the series, $n\cos\theta+(n-1)\cos2\theta+(n-2)\cos3\theta+$, etc.

[*Chrystal's Algebra.*]

69. Proposed by Prof. C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm\ldots\ldots+(\pm1)^n x$, where A, B, C, ...are the binomial coefficients of the $(n+1)^{th}$ order.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

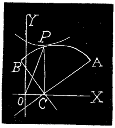
SOLUTIONS OF PROBLEMS.

52. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

If the *center* of a rolling ellipse *move in a horizontal line*, determine the surface on which the ellipse rolls.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let BPA be a quadrant of the ellipse semi-axes AC, and BC, O the position of the center when BC coincides with OY, and $\angle BCP=\theta$. Then



$$PC=y=\frac{ab}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{b}{\sqrt{1-e^2\sin^2\theta}}.$$

$\therefore$ The ellipse rolls on the inner surface of the cylinder

$$y^2+z^2=\frac{b^2}{1-e^2\sin^2\theta}.$$

When $e=0$, this becomes $y^2+z^2=b^2$.

To find the abscissa of the point of contact, we have, since arc $PB=$ arc PG,

$$ds=\sqrt{r^2d\theta^2+dr^2}=\sqrt{y^2d\theta^2+dy^2}\text{ since } PC=r=y;$$

$$\text{also } ds=\sqrt{dx^2+dy^2}.$$

$$\therefore \sqrt{dx^2+dy^2}=\sqrt{y^2d\theta^2+dy^2}.$$

$$\therefore dx=yd\theta,\text{ or } x=\int yd\theta=\int\frac{bd\theta}{\sqrt{1-e^2\sin^2\theta}}=b\mathbf{F}(e,\ \theta).$$

When $e=0$, $x=b\theta$.

[No other solution of this problem was received. EDITOR.]

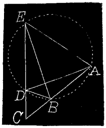
53. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A pole, a certain length of whose top is painted white, is standing on the side of a hill. A person at A observes that the white part of the pole subtends an angle equal to α

and on walking to B, a distance a, directly down the hill towards the foot of the pole the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let DE be the length painted white; then a circle will pass through A, B, D, E. Let $\angle EAD=\angle EBD=\alpha$, $AB=a$, $BC=b$, $\angle DAB=\angle DEB=\theta$, $\angle ABE=\angle ADE=\varphi$, $DC=y$, and $DE=x$.



Then $(x+y)y=(a+b)b$ ....(1).

$AE:a=\sin\varphi:\sin(\alpha+\theta+\varphi)$, $x:AE=\sin\alpha:\sin\varphi$.

$$\therefore x=\frac{a\sin\alpha}{\sin(\alpha+\theta+\varphi)} \quad \text{......................(2)},$$

$b:x+y=\sin\theta:\sin(\alpha+\varphi)$ ....(3),

$(x+y):a+b=\sin(\alpha+\theta):\sin(a+\varphi)$ ....(4).

Eliminating θ between (3) and (4),

$$\left\{\frac{(x+y)^4}{(a+b)^2}-\frac{2b(x+y)^2\cos\alpha}{a+b}+b^2\right\}\sin^2(\alpha+\varphi)=(x+y)^2\sin^2\alpha \quad \text{........(5)}.$$

Eliminating θ between (2) and (3),

$$[\{b^2x^2-x^2(x+y)^2\}^2+4a^2b^2x^2(x+y)^2\sin^2\alpha]\sin^4(\alpha+\varphi)$$
$$-2a^2\sin^2\alpha(x+y)^2\{b^2x^2+x^2(x+y)^2\}$$
$$\sin^2(\alpha+\varphi)+a^4(x+y)^4\sin^4\alpha=0 \quad \text{................(6)}.$$

Eliminating $\sin(\alpha+\varphi)$ between (5) and (6) we get an equation in x and y which with (1) gives us the value of x.

Solved with result in terms of EC by *A. H. HOLMES*, and *FREDERICK R. HONEY*.

PROBLEMS.

58. Proposed by I. J. SCHWATT, Ph. D., Instructor in Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

1. The point of intersection K_a' of the tangent drawn to the circumcircle about the triangle ABC at A and the side BC is harmonic conjugate to K_a with respect to BC. (K_a is the point where the symmedian line through A of the triangle ABC meets the side BC.)

2. The point K_a' is the center of the Apŏllonius circle passing through A of the triangle ABC.

3. Grebes point is on the line joining the middle point of any side of a triangle with the middle point of the altitude to this side.

59. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that the tangent plane at any point of the surface $a^2x^2+b^2y^2+c^2z^2=2bcyz+2acxz+2abxy$ intersects the surface $ayz+bzx+cxy=0$ in two straight lines at right angles to one another.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

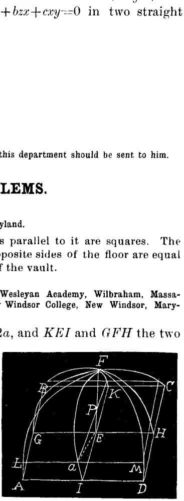
SOLUTIONS OF PROBLEMS.

47. Proposed by Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

The floor of a vault forms a square, and all sections parallel to it are squares. The two vertical sections through the middle points of the opposite sides of the floor are equal semi-circles. Find the convex surface and the volume of the vault.

I. Solution by C. W. M. BLACK, Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts; O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland; A. H. HOLMES, Brunswick, Maine, and the PROPOSER.

Let $ABCD$ represent the base square, side$=2a$, and KEI and GFH the two equal semi-circles, radius$=a$. Let $LMNO$ be another square parallel to the base square, and at the distance $PE=x$ from it. The area of $LMNO$ is$=4(a^2-x^2)$,



$$\therefore\ \text{Vol.}=4\int_0^a(a^2-x^2)dx=\tfrac{8}{3}a^3.$$

Denoting $\angle PEQ$ by θ, we have for the surface

$\int_0^{\frac12\pi}8a\cos\theta d(a\theta)=8a^2$. Or for the volume, $dV=4a^2\cos^2\theta dx$, where x is the vertical distance. $x=a\sin\theta$; $dx=a\cos\theta d\theta$.

$$\therefore\ V=4a^3\int_0^{\frac12\pi}\cos^3\theta d\theta,=a^3\int_0^{\frac12\pi}(\cos3\theta+3\cos\theta)d\theta=\frac{8a^3}{3}.$$

II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

The convex surface of the vault is equivalent to the surface of a right cir-

cular cylinder intercepted by another right circular cylinder, their axes intersecting at right angles, the two cylinders being equal, and the diameter of each equal to that of the vertical sections of the vault.

$\therefore$ Letting the radius$=a$, $S=8a\int_0^a\int_0^{\sqrt{a^2-x^2}}\dfrac{dxdy}{\sqrt{a^2-x^2}}=8a^2$, the equations of the cylinders being $x^2+z^2=a^2$, and $x^2+y^2=a^2$.

The volume is equivalent to that of four wedges cut from the cylinder, $x^2+y^2=a^2$, by the planes, $z=0$, and $z=x$.

$$\therefore\ V=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^x dxdydz=\frac{8a^3}{3}.$$

Also solved by *E. L. SHERWOOD* and *G. B. M. ZERR.*

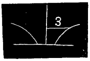
48. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.**

I have a circular section basin 12 inches in perpendicular height; the diameters are as follows: At base, 2 inches; one inch perpendicular height, 6 inches; two inches perpendicular height, 18 inches; three inches perpendicular height, 54 inches; and so on, the diameter being trebled for every inch in height. After a rain the water in the basin is six inches deep, what was the rainfall?

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics, Mississippi Normal College, Houston, Mississippi; C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts, and the PROPOSER.

The basin is generated by revolving the curve $x=3^y$ about the axis of y.

$$\therefore\ \text{Volume of water }=\pi\int_0^6 x^2dy=\pi\int_0^6 3^{2y}dy.$$



$$\therefore\ V=\pi\frac{3^{12}-1}{2\log 3}=\frac{531440\pi}{2\log 3}.$$

Let $x=$depth of rain-fall, then since radius of top of basin$=3^{12}$, $V=\pi 3^{24}x$.

$$\therefore\ x=\frac{565720}{282429536481\log 3}=.00000086 \text{ inches.}$$

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Call x the length of any radius, and y the vertical distance, y being 1 at the bottom of the basin. Then the equation of side of basin is $x=3^{y-1}$,

$$dV=\pi x^2dy,\ V=\pi\int_1^y 3^{2y-2}dy=\frac{\pi[3^{12}-1]}{3\log_e 3}.$$

The radius of upper base$=3^{12}$. Call R the rainfall, then

$$\pi 3.^{24}R=\frac{\pi[3^{12}-1]}{2\log_e 3}. \quad R=\frac{3^{12}-1}{2.3.^{24}\log_e 3}.$$

Also solved by *A. H. HOLMES*, *J. SCHEFFER*, and *B. F. YANNEY*.

ERRATA. In last issue, page 120, line 4 from bottom, for "$\rho=\frac{\theta^2}{c^2}$" read, $\rho^2=\frac{\theta^2}{c^2}$.

PROBLEMS.

55. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A horse is tethered by a rope, a feet long, fastened to a post in a circular fence enclosing a circular piece of ground b feet in diameter. If the horse is outside of the fence over how much ground can he feed? If he is inside the fence over how much ground can he feed? $b>a$ in each case.

56. Proposed by Prof. B. F. BURLESON, Oneida Castle, New York.

Find (1) the length s of the closed curve of the cardioid ; (2) its area A ; (3) if made to revolve about its axis $2a$, find the maximum longitudinal circumference C of the solid generated ; (4) find the surface K of the same ; (5) its volume V ; (6) the distance x_0 of the center of gravity of the solid from the origin O ; and (7) the distance g_0 of the center of gravity of the plane curve from the origin O.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

31. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A perfectly elastic, but perfectly rough mass M, and radius R, rotating in a vertical plane with an angular velocity ω, is let fall from a height, a, upon a perfectly elastic but perfectly rough horizontal plane. Determine the motion of the body after striking the plane. What will be its ultimate motion?

II. Solution by G. B. M ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let V be the vertical velocity of the center just before impact ; u, v, the horizontal and vertical velocities of the center just after the first impact ; ω_1 the

angular velocity after first impact ; u' the velocity of the center just before the second impact ; u_1, ω_2 the values of u, ω_1 just after the second impact, k the radius of gyration.

The equations of motion for first impact are

$$(v+V)(k^2+R^2)=2V(k^2+R^2)\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$u(k^2+R^2)=\omega Rk^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The geometrical condition for no sliding is

$$u-R\omega_1=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

but $V=\sqrt{2ag}$, $k^2=\frac{2}{5}R^2$.

$$\therefore\ v=\sqrt{2ag},\ u=\tfrac{2}{7}R\omega,\ \omega_1=\tfrac{2}{7}\omega,\ u'=\sqrt{v^2+u^2}=\tfrac{1}{7}\sqrt{4\omega^2R^2+98ag}.$$

If β be the angle the center of the sphere makes with the plane just after impact we easily get

$$\cos\beta=\frac{u}{\sqrt{u^2+v^2}}=\frac{u}{u'}=\frac{2R\omega}{\sqrt{4\omega^2R^2+98ag}}.$$

Thus the motion is determined after striking the plane. Let F be the impulse arising from friction, then the equations of motion for second impact are,

$$Mu_1=Mu'\cos\beta+F\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\tfrac{2}{5}MR^2\omega_2=\tfrac{2}{5}MR^2\omega_1-RF\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and the geometrical condition $u_1-R\omega_2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$

$$\therefore\ F/M=-\tfrac{2}{7}(u'\cos\beta-R\omega_1),\ u_1=R\omega_2=\tfrac{5}{7}u'\cos\beta+\tfrac{2}{7}R\omega_1,$$

but $u'\cos\beta=R\omega_1$, $\therefore\ F/M=0$, and no impulsive friction is called into play after the first impact. Hence the center of the sphere describes the same parabola after each impact and the ultimate motion is the same as that after striking the plane.

III. Solution by the PROPOSER.

Each motion of the sphere may be considered, in its reactionary effect, separately. The motion of translation will cause the sphere to rebound after each impact to its original altitude. The time taken to attain the altitude a will be $t=\sqrt{\frac{2a}{g}}$.

The effect of the motion of rotation may be considered in this way : Let a rotating sphere be brought into contact with a plane slowly. The sphere will, of course, roll along the plane. The energy of translation and rotation being equal to the original energy, E, we shall have the same result in the case

under consideration, that is, we shall have the same velocity parallel to the plane, and the same angular velocity as if the sphere were in contact *with* the plane, because there is no slipping at the instance of contact.

Let v_1=velocity parallel to the plane. Then $\frac{v_1}{R}$ = new angular velocity $=\omega_1$.

$$E=\tfrac{1}{5}MR^2\omega^2.$$

$$\text{Energy of translation}=\tfrac{1}{2}Mv_1{}^2.$$

$$\text{New energy of rotation}=\tfrac{1}{5}MR^2\omega_1{}^2=\tfrac{1}{5}Mv_1{}^2.$$

$$\therefore\ \tfrac{1}{5}MR^2\omega^2=\tfrac{1}{2}Mv_1{}^2+\tfrac{1}{5}Mv_1{}^2.\quad \text{Whence,}$$

$$v_1{}^2=\tfrac{2}{7}R^2\omega^2.$$

$$\therefore\ v_1=\sqrt{\tfrac{2}{7}}R\omega, \text{ and}$$

$$\omega_1=\sqrt{\tfrac{2}{7}}\omega.$$

The distance which the sphere will move parallel to the plane while it is attaining its highest altitude will be $=tv_1=2\sqrt{\frac{a}{7g}}R\omega$.

From these data, knowing that the curve will be a parabola, we obtain

$$y^2=-\frac{4R^2\omega^2}{7g}x,$$

the highest point in the origin. The distance between first and second impact is $4\sqrt{a/7g}\,R\omega$. As to the subsequent motion, we have the equation of energy

$$\tfrac{1}{2}Mv_1{}^2+\tfrac{1}{5}Mv_1{}^2=\tfrac{1}{2}Mv_2{}^2+\tfrac{1}{5}Mv_2{}^2,\ \text{or}\ v_2=v$$

and the subsequent parabola will be the same as the first.

PROBLEMS.

37. Proposed by **O. W. ANTHONY, M. Sc.,** Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A thin board of which the elements are given is balanced at the center but inclined at an angle. A sphere of known dimensions is put directly above the point of suspension and liberated. Find the motion of the system. That is, find (a) the time until the sphere leaves the board, (b) the ultimate angular velocity of the board.

38. Proposed by **B. F. FINKEL, A. M.,** Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

A prolate spheriod of revolution is fixed at its focus; a blow is given it at the extremity of the axis minor in a line tangent to the direction perpendicular to the axis major. Find the axis about which the body begins to rotate. [From *Loudon's Rigid Dynamics.*]

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

37. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Find the first four integral values of n in $\dfrac{n(5n-3)}{2}=\square$.

I. Solution by the PROPOSER, and Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

Let the heptagonal numbers $\dfrac{n(5n-3)}{2}=\square=y^2$. Clearing of fractions, then multiplying by 20 and adding 9 to both sides, $(10n-3)^2=40y^2+9=\square=x^2$. $\therefore n=(x+3)/10$........(1). Let $x^2-40y^2=9$ be written $3^2x_1{}^2-40.3^2y_1{}^2=3^2$. Dividing by 3^2 and solving $x_1{}^2-40y_1{}^2=1$, the convergent of $\sqrt{40}$ is $19/3$. $\therefore x_1=19$; by the general formula $x_{n+1}=2x_1\times x_n-x_{n-1}$, we have $x_1=1$, 19, 721, 27379, 1039681, 39, 480, 499, etc. As $x=3x_1$ and as integral values for n can only be obtained by the numbers ending in 9, then in (1) $n=1$, 6, 8214, and 11844150.

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

The expression readily reduces to $10n^2-6n=\square$......(1). It is readily seen that $n=1$ satisfies this equation. Take $n=m+1$, substitute it in (1), reduce and we have $10m^2+14m+4=\square=(\text{say})\ (pm-2)^2$, from which we obtain $m=(4p+14)/(p^2-10)$. Take $p=4$ and we have $m=5$, and $n=6$, the second value. Now take $n=m+6$, substitute in (1) and reduce as before and we find, $m=43$, and $n=49$, the third value. In $(4p+14)/(p^2-10)$ take $p=19/6$, $p^2-10=1/36$ and we have $m=960$, and $n=961$, the fourth value.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

If we put the expression equal x^2 and reduce, we readily obtain $10n-3\pm\sqrt{40x^2+9}$. Putting $x=1$, 2, 9, 40 and 77, respectively, I find the first four integral values of n to be, respectively, ±1, 6, -25, and 49.

38. Proposed by H. C. WILKES, Skull Run, West Virginia.

Let n be any number and let $n^3+1=x$.
Then $x^3+(2x-3)^3+(nx-3n)^3=n^3x^3$. Demonstrate.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

The simplest way is to substitute the value of x and expand. An identity is the result.

II. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Substituting n^3+1 for x, $(n^3+1)^3+(2n^3-1)^3+(n^4-2n)^3=(n^4+n)^3$,

which, if we put c for n, is the same as equation (12) on page 155 of Vol. II., No. 9 of the *Mathematical Magazine*, and is an identity as will be found by performing the indicated operations and adding.

III. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

Suppose the statement true : $x^3+(2x-3)^3+(nx-3n)^3=n^3x^3$.

Then, $x^3+8x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3x-27n^3=n^3x^3$.

Whence, $(x-1)(x^2-3x+3)-n^3(x^2-3x+3)=0$. Whence, $x-1-n^3=0$.

Whence, $n^3+1=x$, which is the hypothesis. $\therefore$ The above supposition is true.

IV. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tennessee.

Performing the operations we have

$$x^3+8x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3x-27n^3, \text{ or}$$

$$9x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3(x-1), \text{ but } n^3=x-1\ ;$$

hence $9x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27(x-1)^2$, which upon reduction gives

$$9x^3-9x^2-9n^3x^2+n^3x^3,=9x^3-9x^2-9x^2(x-1)+n^3x^3,$$

$$=9x^3-9x^2-9x^3+9x^2+n^3x_3,\ =n_3x_3.$$

Also solved by *J. H. DRUMMOND, M. A. GRUBER, J. SCHEFFER*, and *G. B. M. ZERR*.

39. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The m^{th} root of the n^{th} power of an *integral* number is a perfect p^{th} power. What is the number ?

Solutions by J. H. DRUMMOND, LL. D., Portland, Maine ; M. A. GRUBER, A. M., Washington, D. C., and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $x^{1\div m}=a^p$, then $x=a^{pm}$, in which a may be any integral number : for the n^{th} power of a^{pm} must also be a p^{th} power. [J. H. DRUMMOND.]

Manifestly any whole number raised to the mp^{th} power.

[J. SCHEFFER.]

Let x=the integral number. Then $x^{n\div m}=a^p$. Raising to m^{th} power and extracting n^{th} root, we obtain $x=a^{mp\div n}$, or $\sqrt[n]{a^{mp}}$. $\therefore$ The required integral number is the n^{th} root of the mp^{th} power of any integer, mp being a multiple of n.

[M. A. GRUBER.]

Also solved by *G. B. M. ZERR*.

PROBLEMS.

45. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six sets of values in which the sum of two consecutive integral squares equals a square.

46. Proposed by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

If any positive integral number N be divided by another positive integral number D, leaving a remainder of 1, then any positive integral power of N, divided by D, will leave a remainder of 1.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

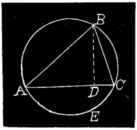
SOLUTIONS OF PROBLEMS.

28. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Missouri.

What is the average area of all triangles having a given base, b, and a given vertical angle, α?

Solution by the PROPOSER.

Let ABC be a triangle whose base $AC=b$ and vertical angle $ABC=\alpha$. Let $BC=x$, $\angle BAC=\theta$, and Δ average area required.



Then $x=\dfrac{b}{\sin\alpha}\sin\theta$, and $BD=x\sin\angle BCA$

$=\dfrac{b}{\sin\alpha}\sin\theta\sin(\theta+\alpha)$.

$\therefore$ Area of the triangle $=\dfrac{b^2}{2\sin\alpha}\sin\theta\sin(\theta+\alpha)$.

The limits of θ are 0 and $\pi-\alpha$.

$$\therefore \Delta=\frac{\int_0^{\pi-\alpha}\frac{b^2}{2\sin\alpha}\sin\theta\sin(\theta+\alpha)d\theta}{\int_0^{\pi-\alpha}d\theta}=\frac{b^2}{2(\pi-\alpha)\sin\alpha}\int_0^{\pi-\alpha}\sin\theta\sin(\theta+\alpha)d\theta$$

$$=\frac{b^2}{2(\pi-\alpha)\sin\alpha}\Big[(-\tfrac{1}{2}\sin\theta\cos\theta+\tfrac{1}{2}\theta(\cos\alpha+\tfrac{1}{2}\sin^2\theta\sin\alpha\Big]_0^{\pi-\alpha}$$

$$=\frac{b^2}{4(\pi-\alpha)}\Big\{(\pi-\alpha)\cot\alpha+1\Big\}.$$

COROLLARY. Let $\alpha=\frac{1}{2}\pi$; then $\Delta=\frac{b^2}{2\pi}$, the same as problem 26.

[NOTE.—By mistake in numbering the problems in this department, number 28 was omitted. The above problem and solution are inserted that problems be numbered consecutively. EDITOR.]

29. Proposed by JOHN DOLMAN, Jr., Philadelphia, Pennsylvania.

Neglecting perturbations, what is the average distance of the earth from the sun?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The focus being the pole, the polar equation to the ellipse is

$$r=\frac{a(1-e^2)}{1-e\cos\theta}\dots\dots(1).$$

I. The radii vectores being drawn at equal angular intervals,

$$m'=\frac{\int r d\theta}{\int d\theta}=a(1-e^2)\frac{\int_0^\pi \frac{d\theta}{1-e\cos\theta}}{\int_0^\pi d\theta}=a\sqrt{1-e^2}=b.$$

II. If x be the abscissa of any point on the curve, the focal distance is

$$r=a-ex\dots\dots(2),$$

$$\text{and } m''=\frac{\int_a^{+a}(a-ex)dx}{\int_{-a}^{+a}dx}=a,$$

the points on the curve being so taken that their abscissas increase uniformly.

III. If the number of radii vectores depends upon the length of the curve,

$$m'''=\frac{\int r ds}{\int ds},$$

ds being an element of the curve.

Also solved as I. above by *Profs. F. P. MATZ*, and *O. W. ANTHONY*, and as III. by *Prof. G. B. M. ZERR*.

PROBLEMS.

37. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

Required the average area of all triangles two of whose sides are a and b.

38. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two arrows are sticking in a circular target: show that the chance that their distance is greater than the radius of the target is $3\sqrt{3}/4\pi$. [From *Todhunter's Integral Calculus*, page 335.]

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

33. Proposed by Prof. ALEXANDER ROSS, C. E., Sebastopol, California.

From a point P without a square field ABC, the distances PA, PB, and PC measured to the corners are, respectively, 70, 40, and 60 chains. What is the area of the field ?

I. Solution by A. H. BELL, Hillsboro, Illinois, and A. H. HOLMES, Brunswick, Maine.

Let $a>b>c$ equal the distances 70, 60, and 40, and let $x=a$ side of the square field. Then $\cos A=\dfrac{a^2+x^2-c^2}{2ax}$, and this multiplied by $a=AF=\dfrac{a^2+x^2-c^2}{2x}$. $AF-AB=BF=EP=\dfrac{a^2-c^2-x^2}{2x}$; then, also, $BE=\dfrac{b^2-c^2-x^2}{2x}$.

$$\therefore\ \frac{(a^2-c^2-x^2)^2+(b^2-c^2-x^2)^2}{4x^2}=c^2\ \dots\dots\dots\dots\dots(1).$$

$$x^4-(a^2+b^2)x^2=c^2(a^2+b^2)-\frac{a^4+b^4+2c^4}{2}\ \dots\dots\dots(2).$$

$$\text{Area of square}=x^2=\tfrac{1}{2}\left[a^2+b^2\pm\sqrt{4c^2(a^2+b^2-c^2)-(a^2-b^2)^2}\right]\ \dots.(3).$$

Then area required $=(8500\pm6516.901)\div2=750.84\frac{1}{2}$ or 99.155 acres.

The second is the value required; the other is for point within the field.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let $ABCD$ be the square, $OA=70=a$, $OB=40=c$, $OC=60=b$, O the origin, (x, y) co-ordinates of A, (u, v) co-ordinates of C, $\angle ABE=\theta$, $\angle EBC=\frac{\pi}{2}-\theta$.

$$\therefore\ (x-c)^2+y^2=(u-c)^2+v^2,\ x^2+y^2=a^2,\ u^2+v^2=b^2\ \dots\ \dots\dots(1, 2, 3).$$

$\tan\theta=\dfrac{y}{x-c}$, $\cot\theta=\dfrac{v}{u-c}$, $\therefore \dfrac{y}{x-c}=\dfrac{u-c}{v}$. $\therefore yv=(x-c)(u-c)$..........(4.)

(2) and (3) in (1) and (4) gives, $2c(x-u)=a^2-b^2$, ...(5),

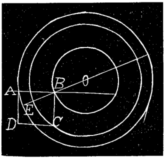
$(a^2-x^2)(b^2-u^2)=(x-c)^2(u-c)^2$........................(6). (5) in (6) gives

$\{4a^2c^2-(a^2-b^2)^2-4cu(a^2-b^2)-4c^2u^2\}\ (b^2-u^2)=(a^2-b^2+2uc-2c^2)\ (u-c)^2$.

$\therefore (74175-520u-16u^2)(3600-u^2)-(4u-95)^2(u-40)^2$.

$\therefore 32u^3-2840u^2+825u+3157375=0$. Let $u=z+\frac{355}{12}$.

$\therefore z^3-\frac{249575}{96}z+\frac{164676875}{3456}=0$.



This equation has three roots.

$\therefore z_1=23.02208$, $z_2=36.23197$, $z_3=-58.43863$.

$\therefore u_1=52.60541$, $u_2=65.81530$, $u_3=-28.85530$.

$\therefore x_1=69.35541$, $x_2=82.56530$, $x_3=-12.10530$.

$\therefore y_1=9.47772$, $y_3=68.94535$.

The first values satisfy the problem in question; the second must be rejected as not admissible; while the third values satisfy the problem for the point within the field.

$\therefore$ area $ABCD=(x-c)^2+y^2=951.5672$ square chains$=95.15672$ acres.

When the point is within field, area$=(x-c)^2+y^2=7468.424$ square chains$=746.8424$ acres.

Also solved by *O. W. ANTHONY*.

34. **Proposed by THOS. U. TAYLOR, C. E., M. C., Department of Engineering, University of Texas, Austin, Texas.**

Given a variable parallelogrom $ABCP$, where P remains fixed. A moves on an irregular plane curve (closed) and C moves on another irregular plane curve (closed) whose plane is parallel to the plane of (A) curve. The generator PC moves completely around and returns to its initial position, AB always moving parallel to PC, and, of course, returns to its initial position. If distance between planes (A) and $(C)=h$, show by elementary mathematics and without using theorem of Koppe that volume of solid generated by variable parallelogram $ABCP=\frac{1}{2}h$ (area generated by $AP+$area generated by BC).

No solution of this problem has been received.

PROBLEMS.

38. **Proposed by F. M. PRIEST, Mona House, St. Louis, Missouri.**

Suppose two cylindrical iron shafts, each 6 inches in diameter and respectively, 20 and 40 feet in height, are both standing perpendicular at the sea level. They start to fall in still air, how long will it require each one to fall to a horizontal position ?

39. Proposed by **WILLIAM SYMMONDS, A. M.,** Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

A straight inflexible bar of uniform weight and thickness, length m is suspended at the two ends by a string without weight, length $l>m$ passing freely over a peg driven in a perpendicular wall. Describe and analyze the curve traced on the wall by the ends of the hanging bar.

NOTE.—Problem No. 43, Calculus, should read as suggested by Dr. E. A. Bowser on page 60 of February number. Prof. Black had noted the correct form in his copy of Williamson, and so sent it to the MONTHLY, but an error was made in printing the expression. A letter from Dr. Williamson, Trinity College, Dublin, Ireland, acknowledges the error in his work, and says it will be corrected in the forthcoming new edition of his book.

WANTED.—Some one to give a list, partial or complete, of the curves of the fourth degree that have received particular names, such as the "Lemniscate," "Cocked Hat," "Devil's Walking Stick," "Conchoid," etc.

COOPER D. SCHMITT.

BOOKS.

Warren Colburn's First Lessons : Intellectual Arithmetic upon the Inductive Method of Instruction (1891).

H. N. Wheeler's Second Lessons (1893). Boston, New York, and Chicago : Houghton, Mifflin and Company.

First Lessons, which has been famous for three-fourths of a century, contains, besides the four fundamental operations, little but fractions and denominate numbers. It has no rules and but little written work. It follows the inductive method—the method of "Practice before Theory"—which is based on the soundest psychological principles. This book should be in the hands of every teacher, whether used as a class book or not.

Wheeler's Second Lessons is intended as a continuation of Colburn's First Lessons, and is well adapted for that purpose. J. M. C.

Logarithmic Tables. By Professor George William Jones, of Cornell University. Sixth Edition. Royal 8vo. Cloth. 160 pages. Price, $1.00. Published by the author.

These are the best tables that we have yet seen. Eighteen tables (four-place, six-place, ten-place) with full explanation for their use, for use in the class-room, laboratory, and the office. The tables of Mathematical Constants, Chemistry, Engineering, and Physics deserve special mention. Also Table IX which gives the prime factors of composite numbers less than 20000, and Tables X and XI which give the squares and cubes of all

three figure numbers in full. If you want a complete and valuable set of tables buy a copy of Prof. Jones, and you will need none other. B. F. F.

Mathematical Papers Read at the International Mathematical Congress held in Connection with the Columbian Exposition, Chicago, 1893. Edited by the Committee of the Congress, E. Hastings Moore, Oskar Bolza, Heinrich Maschke, Henry S. White. Large 8vo. Cloth, 412 pages. Price, $4.00, New York: Macmillan & Co.

This important collection of important mathematical papers is given to the mathematicians of all time at no small amount of labor at the hands of the editors.

It is especially fitting that these papers, many of which indicate the high-water mark of the development of mathematics at the present time, should be collected and bound for the benefit of the mathematicians of the centuries yet to be.

Neither the management of the Exposition nor the government of the United States had made any provisions for the publication of the proceedings of any of the Chicago Congresses. No publisher was found willing to issue the papers at his own risk.

At last a guarantee fund of one thousand dollars in all was subscribed, six hundred dollars by the American Mathematical Society, and four hundred dollars by members of the Society and olher mathematicians. On the basis of this guaranty fund the publication of the volume of the papers was made possible, the American Mathematical Society assuming the financial, and the Chicago Committee the editorial responsibility. *Preface.* B. F. F.

NOTES.

Dr. William B. Smith, of the Tulane University of Louisiana, has in press the first volume of his Infinitisimal Analysis.

The June number of the MONTHLY will be mailed about the 16th of the month. In this issue will appear the biography of Mr. W. J. C. Miller.

Dr. George Bruce Halsted, of the University of Texas, and Dr. David E. Smith, of the Michigan State Normal School, will spend the summer in Europe. Dr. Halsted will visit Paris, Genoa, Buda Pest, Moskow, Kazan, etc.

ERRATA. In Prof. G. B. M. Zerr's paper, "The Centroid of Areas and Volumes," in value of $\bar{x}$, bottom of page 73, in numerator read $\frac{1}{2}(2p+1)$ for "$\frac{1}{2})2p+1)$," and in denominator read $\frac{k}{2}(2n+1)$ for "$\frac{k}{2}(2n=1)$" and $\frac{h}{2}(2m+1)$ for "$\frac{h}{3}(2m+1)$. Page 75, line 8 read $\left(\frac{z}{c}\right)$ for "$\frac{z}{c}$." Page 102, last line, read $-a^4\log\left(\frac{a^2+4h+2\sqrt{2a^2h+4h^2}}{a^2}\right)$ for "$-a^4\log\left(\frac{a^2+4h+\sqrt{2a^2h+4h^2}}{a^2}\right)$," Page 103, first line, last expression in numerator read $\sqrt{2a^2h+4h^2}$ for "$\sqrt{2a^2h+4h^2}$" and in second line read dx for "bx."